OLIMPIADA NAŢIONALĂ DE MATEMATICĂ

Faza locală-14.02.2014

Clasa a XI-a

Soluţii

1) a) $X=\begin{pmatrix} a & b \\ c & d \end{pmatrix}, Y=\begin{pmatrix} m & n \\ p & q \end{pmatrix}; a,b,c,d,m,n,p,q\in R$.

$$\det(X+Y)+\det(X-Y)=\begin{vmatrix} a+m & b+n \\ c+p & d+q \end{vmatrix}+\begin{vmatrix} a-m & b-n \\ c-p & d-q \end{vmatrix}=$$

$$=ad+aq+dm+mq-bc-bp-cn-np+ad-aq-dm+mq-bc+bp+cn-np=2(ad+mq-bc-np)$$

$$2\det(X)+2\det(Y)=2\begin{vmatrix} a & b \\ c & d \end{vmatrix}+2\begin{vmatrix} m & n \\ p & q \end{vmatrix}=2(ad-bc+mq-np)$$

Prin urmare $\det(X+Y)+\det(X-Y)=2\det(X)+2\det(Y)$.

b) Folosind a) obţinem

$$2\det\left(X^2+Y^2\right)+2\det(XY+YX)=\det\left(X^2+Y^2+XY+YX\right)+\det\left(X^2+Y^2-XY-YX\right)=$$

$$=\det\left[(X+Y)^2\right]+\det\left[(X-Y)^2\right]=\left[\det(X+Y)\right]^2+\left[\det(X-Y)\right]^2\geq 0\Rightarrow$$

$$\Rightarrow \det\left(X^2+Y^2\right)\geq -\det(XY+YX)\geq 0.$$

2) a) $\lim\limits_{x\to 0}\left(e^x+\sin 2x\right)^{\frac{1}{x}}\overset{1^\infty}{=}\lim\limits_{x\to 0}\left\{\left[1+\left(e^x+\sin 2x-1\right)\right]^{\frac{1}{e^x+\sin 2x-1}}\right\}^{\frac{e^x+\sin 2x-1}{x}}=$

$$=e^{\lim\limits_{x\to 0}\left(\frac{e^x-1}{x}+\frac{\sin 2x}{2x}\cdot 2\right)}=e^3.$$

b) $\lim\limits_{x\to 1}\left(\dfrac{a}{1-x^6}-\dfrac{b}{1-x^9}\right)=\lim\limits_{x\to 1}\dfrac{1}{1-x^3}\left(\dfrac{a}{1+x^3}-\dfrac{b}{1+x^3+x^6}\right)=\dfrac{\frac{a}{2}-\frac{b}{3}}{0}$

Dacă $\dfrac{a}{2}-\dfrac{b}{3}\neq 0$, atunci limita este infinită şi nu convine.

Dacă $\dfrac{a}{2}-\dfrac{b}{3}=0\Rightarrow b=\dfrac{3a}{2}$.

$$\lim_{x\to 1}\left(\frac{a}{1-x^6}-\frac{b}{1-x^9}\right)=a\lim_{x\to 1}\frac{1}{1-x^3}\left(\frac{1}{1+x^3}-\frac{3}{2\left(1+x^3+x^6\right)}\right)=a\lim_{x\to 1}\frac{1}{1-x^3}\cdot\frac{2x^6-x^3-1}{2\left(1+x^3\right)\left(1+x^3+x^6\right)}=$$

$$=a\lim_{x\to 1}\frac{1}{1-x^3}\cdot\frac{\left(2x^3+1\right)\left(x^3-1\right)}{2\left(1+x^3\right)\left(1+x^3+x^6\right)}=-\frac{a}{4}=-\frac{1}{2}\Rightarrow a=2,\ b=3.$$

3) a) $X=\begin{pmatrix} a & b \\ c & d \end{pmatrix}; a,b,c,d\in R$. $X^{2014}=X^{2013}\cdot X=X\cdot X^{2013}$, adică

$$\begin{pmatrix} 1 & 2014 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 2014 \\ 0 & 1 \end{pmatrix}\Rightarrow\begin{cases} a+2014c=a \\ b+2014d=2014a+b \\ c=c \\ d=2014c+d \end{cases}\Rightarrow$$

$$\Rightarrow\begin{cases} c=0 \\ d=a \end{cases}\Rightarrow X=\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}=aI_2+B,\ B=\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix},\ (aI_2)B=B(aI_2).$$

$B^2=O_2$ şi $B^n=O_2, \forall n\geq 2.$

$$X^{2013}=(aI_2+B)^{2013}=a^{2013}I_2+2013a^{2012}B=\begin{pmatrix} a^{2013} & 2013a^{2012}b \\ 0 & a^{2013} \end{pmatrix}=\begin{pmatrix} 1 & 2014 \\ 0 & 1 \end{pmatrix}$$

Se obţine sistemul $\begin{cases} a^{2013}=1 \\ 2013a^{2012}b=2014 \end{cases}$ cu soluţiile reale $\begin{cases} a=1 \\ b=\dfrac{2014}{2013} \end{cases}\Rightarrow X=\begin{pmatrix} 1 & \dfrac{2014}{2013} \\ 0 & 1 \end{pmatrix}$

b) Cum $A^2-tr(A)\cdot A+\det(A)\cdot I_2=O_2\Rightarrow A^2=-\det(A)\cdot I_2\Rightarrow tr(A^2)=-2\det(A)=0\Rightarrow\det(A)=0$

$A^2=O_2\Rightarrow A^{2014}=O_2.$

4) Şirul $(a_n)_{n\geq 1}$ are termenii strict pozitivi. Cum $\dfrac{a_{n+1}}{a_n}=\dfrac{1}{1+na_n}\leq 1$, şirul este convergent.

Fie $l=\lim\limits_{n\to\infty}a_n$. Dacă $l>0$, atunci $\lim\limits_{n\to\infty}(1+na_n)=\infty$ şi $\lim\limits_{n\to\infty}a_{n+1}=\lim\limits_{n\to\infty}\dfrac{a_n}{1+na_n}=0$, adică $l=0$, fals. Prin urmare $\lim\limits_{n\to\infty}a_n=0$.

Aplicând succesiv lema Stolz-Cesaro, avem:

$$\lim_{n\to\infty}\frac{1}{n^4}\left(\frac{1}{a_1}+\frac{2}{a_2}+\ldots+\frac{n}{a_n}\right)=\lim_{n\to\infty}\frac{\frac{n}{a_n}}{n^4-(n-1)^4}=\lim_{n\to\infty}\frac{\frac{n}{a_n}}{4n^3-6n^2+4n-1}=\lim_{n\to\infty}\frac{\frac{1}{a_n}}{n^2}\cdot\frac{n^3}{4n^3-6n^2+4n-1}=$$

$$=\frac{1}{4}\lim_{n\to\infty}\frac{\frac{1}{a_n}}{n^2}=\frac{1}{4}\lim_{n\to\infty}\frac{\frac{1}{a_{n+1}}-\frac{1}{a_n}}{(n+1)^2-n^2}=\frac{1}{4}\lim_{n\to\infty}\frac{\frac{1+na_n-1}{a_n}}{2n+1}=\frac{1}{8}.$$

OLIMPIADA NAŢIONALĂ DE MATEMATICĂ

Faza locală-15.02.2014

Clasa a XI-a

Bareme

1) a) $X=\begin{pmatrix} a & b \\ c & d \end{pmatrix}, Y=\begin{pmatrix} m & n \\ p & q \end{pmatrix}; a,b,c,d,m,n,p,q\in R$.

$\det(X+Y)+\det(X-Y)=\begin{vmatrix} a+m & b+n \\ c+p & d+q \end{vmatrix}+\begin{vmatrix} a-m & b-n \\ c-p & d-q \end{vmatrix}=ad+aq+dm+mq-$

$-bc-bp-cn-np+ad-aq-dm+mq-bc+bp+cn-np=2(ad+mq-bc-np)$......................2p

$2\det(X)+2\det(Y)=2\begin{vmatrix} a & b \\ c & d \end{vmatrix}+2\begin{vmatrix} m & n \\ p & q \end{vmatrix}=2(ad-bc+mq-np)$ 1p

Prin urmare $\det(X+Y)+\det(X-Y)=2\det(X)+2\det(Y)$.

b) Folosind a) obţinem

$2\det(X^2+Y^2)+2\det(XY+YX)=\det(X^2+Y^2+XY+YX)+\det(X^2+Y^2-XY-YX)=$.........1p

$=\det\left[(X+Y)^2\right]+\det\left[(X-Y)^2\right]=$..1p

$=\left[\det(X+Y)\right]^2+\left[\det(X-Y)\right]^2\geq 0\Rightarrow$ 1p

$\Rightarrow\det(X^2+Y^2)\geq-\det(XY+YX)\geq 0$...1p

2) a) $\lim\limits_{x\to 0}\left(e^x+\sin 2x\right)^{\frac{1}{x}}\overset{1^\infty}{=}\lim\limits_{x\to 0}\left\{\left[1+\left(e^x+\sin 2x-1\right)\right]^{\frac{1}{e^x+\sin 2x-1}}\right\}^{\frac{e^x+\sin 2x-1}{x}}=$ ………………..……….1p

$=e^{\lim\limits_{x\to 0}\left(\frac{e^x-1}{x}+\frac{\sin 2x}{2x}\cdot 2\right)}=$ ……………………………………………………………………..……….1p

$=e^3$.……………………………………………………………………………...………1p

b) $\lim\limits_{x\to 1}\left(\frac{a}{1-x^6}-\frac{b}{1-x^9}\right)=\lim\limits_{x\to 1}\frac{1}{1-x^3}\left(\frac{a}{1+x^3}-\frac{b}{1+x^3+x^6}\right)=\frac{\frac{a}{2}-\frac{b}{3}}{0}$ 1p

Dacă $\frac{a}{2}-\frac{b}{3}\neq 0$, atunci limita este infinită şi nu convine.

Dacă $\frac{a}{2}-\frac{b}{3}=0\Rightarrow b=\frac{3a}{2}$ 1p

$\lim\limits_{x\to 1}\left(\frac{a}{1-x^6}-\frac{b}{1-x^9}\right)=a\lim\limits_{x\to 1}\frac{1}{1-x^3}\left(\frac{1}{1+x^3}-\frac{3}{2(1+x^3+x^6)}\right)=a\lim\limits_{x\to 1}\frac{1}{1-x^3}\cdot\frac{2x^6-x^3-1}{2(1+x^3)(1+x^3+x^6)}=$

$$=a\lim_{x\to 1}\frac{1}{1-x^3}\cdot\frac{\left(2x^3+1\right)\left(x^3-1\right)}{2\left(1+x^3\right)\left(1+x^3+x^6\right)}=$$ 1p

$$=-\frac{a}{4}=-\frac{1}{2}\Rightarrow a=2,\ b=3.$$ 1p

3) a) $X=\begin{pmatrix} a & b \\ c & d \end{pmatrix}; a,b,c,d\in R.\ X^{2014}=X^{2013}\cdot X=X\cdot X^{2013}$, adică

$$\begin{pmatrix} 1 & 2014 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 2014 \\ 0 & 1 \end{pmatrix}\Rightarrow\begin{cases} a+2014c=a \\ b+2014d=2014a+b \\ c=c \\ d=2014c+d \end{cases}\Rightarrow$$

$$\Rightarrow\begin{cases} c=0 \\ d=a \end{cases}\Rightarrow X=\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}=$$ 1p

$$=aI_2+B,\ B=\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix},\ (aI_2)B=B(aI_2).$$

$B^2=O_2$ şi $B^n=O_2,\ \forall n\geq 2.$

$$X^{2013}=\left(aI_2+B\right)^{2013}=a^{2013}I_2+2013a^{2012}B=\begin{pmatrix} a^{2013} & 2013a^{2012}b \\ 0 & a^{2013} \end{pmatrix}=\begin{pmatrix} 1 & 2014 \\ 0 & 1 \end{pmatrix}$$ 2p

Se obţine sistemul $\begin{cases} a^{2013}=1 \\ 2013a^{2012}b=2014 \end{cases}$ cu soluţiile reale

$$\begin{cases} a=1 \\ b=\dfrac{2014}{2013} \end{cases}\Rightarrow X=\begin{pmatrix} 1 & \dfrac{2014}{2013} \\ 0 & 1 \end{pmatrix}$$ 1p

b) Cum $A^2-tr(A)\cdot A+\det(A)\cdot I_2=O_2\Rightarrow A^2=-\det(A)\cdot I_2\Rightarrow$ 1p

$\Rightarrow tr(A^2)=-2\det(A)=0\Rightarrow\det(A)=0$ 1p

$A^2=O_2\Rightarrow A^{2014}=O_2.$ 1p

4) Şirul $(a_n)_{n\geq 1}$ are termenii strict pozitivi. Cum $\frac{a_{n+1}}{a_n}=\frac{1}{1+na_n}\leq 1,$ şirul este convergent...........1p

Fie $l=\lim\limits_{n\to\infty} a_n$. Dacă $l>0,$ atunci $\lim\limits_{n\to\infty}(1+na_n)=\infty$ şi $\lim\limits_{n\to\infty} a_{n+1}=\lim\limits_{n\to\infty}\frac{a_n}{1+na_n}=0,$ adică $l=0,$ fals. Prin urmare $\lim\limits_{n\to\infty} a_n=0.$ 1p

Aplicând succesiv lema Stolz-Cesaro, avem:

$$\lim_{n\to\infty}\frac{1}{n^4}\left(\frac{1}{a_1}+\frac{2}{a_2}+\ldots+\frac{n}{a_n}\right)=\lim_{n\to\infty}\frac{\frac{n}{a_n}}{n^4-(n-1)^4}=\lim_{n\to\infty}\frac{\frac{n}{a_n}}{4n^3-6n^2+4n-1}= \text{..2p}$$

$$=\lim_{n\to\infty}\frac{\frac{1}{a_n}}{n^2}\cdot\frac{n^3}{4n^3-6n^2+4n-1}=\frac{1}{4}\lim_{n\to\infty}\frac{\frac{1}{a_n}}{n^2}= \text{..1p}$$

$$=\frac{1}{4}\lim_{n\to\infty}\frac{\frac{1}{a_{n+1}}-\frac{1}{a_n}}{(n+1)^2-n^2}=\frac{1}{4}\lim_{n\to\infty}\frac{\frac{1+na_n-1}{a_n}}{2n+1}=\frac{1}{8}. \text{..2p}$$