OLIMPIADA DE MATEMATICA
ETAPA LOCALA
16 februarie 2013

Clasa a XI-a

Barem de notare :

Subiectul I Fie matricea de ordin n$\in$ N*, $A_n = (a_{ij})_{i,j=\overline{1,n}}$, unde $a_{ii} = i$,i=$\overline{1, n}$, $a_{i,i+1} = a_{i+1,i} = 1, i = \overline{1, n-1}$ si $a_{ij} = 0$ in rest.Daca $\Delta_n = \det(A_n)$, se cere :

a) Aratati ca $\Delta_n = n \cdot \Delta_{n-1} - \Delta_{n-2}, \forall n \geq 3$;

b) Calculati Δ_n , n≥ 1 .

Rezolvare : a)) $\Delta_2 = 1$, $\Delta_3 = 2$, $\Delta_4 = 7$ (calcul direct)……………………….……………………3p

b)Dezvoltă Δ_n dupa coloana n şi obţine:

$$\Delta_n = \begin{vmatrix} 1\ 1\ 0\ 0\ 0 \ldots 0 \\ 1\ 2\ 1\ 0\ 0 \ldots 0 \\ 0\ 1\ 3\ 1\ 0 \ldots 0 \\ \ldots \ldots \ldots \ldots \ldots . \\ 0\ 0\ 0\ 0\ 0 \ldots 1\ n \end{vmatrix} = (-1)^{2n-1} \cdot \begin{vmatrix} 1\ 1\ 0\ 0\ 0 \ldots 0 \\ 1\ 2\ 1\ 0\ 0 \ldots 0 \\ 0\ 1\ 3\ 1\ 0 \ldots 0 \\ \ldots \ldots \ldots \ldots \ldots \ldots \\ 0\ 0\ 0\ 0\ 0 \ldots 1 \end{vmatrix} + (-1)^{2n} \begin{vmatrix} 1\ 1\ 0\ 0\ 0 \ldots 0 \\ 1\ 2\ 1\ 0\ 0 \ldots 0 \\ 0\ 1\ 3\ 1\ 0 \ldots 0 \\ \ldots \ldots \ldots \ldots \ldots \ldots \\ 0\ 0\ 0\ 0\ 0 \ldots n \end{vmatrix} = n \cdot \Delta_{n-1} - \Delta_{n-1}$$

(cei doi determinant se dezvoltă după ultima coloană) …………………………….. …4p

Subiectul II(7p)

Rezolvati in $M_2(R)$ ecuatia : $X^3 + 3X^2 + 3X = \begin{pmatrix} 7 & -8052 \\ 0 & 7 \end{pmatrix}$.

Rezolvare : Adunăm I_2 in ambii membrii ai ecuatiei, notăm Y=X+I_2 şi obţinem ecuaţia

$Y^3 = \begin{pmatrix} 8 & -8052 \\ 0 & 8 \end{pmatrix} = A$ (se ţine cont că X$\cdot I_2 = I_2 \cdot X = X$)…………………………...…….2p

$Y^4 = A \cdot Y = Y \cdot A$, de unde se obţine Y=$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$………………..……………………………2p

Calculeaza $Y^3 = \begin{pmatrix} a^3 & a^2b(1+2a) \\ 0 & a^3 \end{pmatrix}$=$\begin{pmatrix} 8 & -8052 \\ 0 & 8 \end{pmatrix}$………………………………………1p

Gaseste Y=$\begin{pmatrix} 2 & -2013 \\ 0 & 2 \end{pmatrix}$ si apoi X=$\begin{pmatrix} 1 & -2013 \\ 0 & 1 \end{pmatrix}$ ……………………………………...…2p

Subiectul III(7p)

Fie sirul $(x_n)_{n\in N}$ definit prin $x_{n+1} = x_n + e^{-x_n}$ $\forall n \in N$ si $x_0 \in \mathbf{R}$. Sa se arate ca

$$\lim_{n\to\infty} \frac{\ln n}{x_n} = 1$$

Rezolvare: Monotonia : x_{n+1}-x_n=$e^{-x_n} > 0$, $x_n -$ strict crescator………………………1p

Marginirea: Presupunem x_n marginit superior , atunci din Teorema lui Weierstrass obtinem x_n convergent , si trecand la limita in relatia de recurenta se obtine l=l+e^{-l} cu l$\in R$, deci $\lim_{n\to\infty} x_n = +\infty$ ………………………………………………………………………..2p

Aplicand repetat Lema Cesaro-Stoltz obtinem $\lim_{n\to\infty} \frac{lnn}{x_n} = \lim_{n\to\infty} \frac{ln\left(\frac{n+1}{n}\right)}{x_{n+1}-x_n}$=$\lim_{n\to\infty} e^{x_n} \cdot ln\left(\frac{n+1}{n}\right)$=$\lim_{n\to\infty} \frac{e^{x_n}}{n} \cdot \lim_{n\to\infty} n \cdot ln\left(\frac{n+1}{n}\right)$ =$\lim_{n\to\infty} \frac{e^{x_{n+1}}-e^{x_n}}{n+1-n}$=$\lim_{n\to\infty} e^{x_n} \cdot (e^{x_{n+1}-x_n} - 1)$ =$\lim_{n\to\infty} \frac{(e^{x_{n+1}-x_n}-1)}{x_{n+1}-x_n}$=1 ……..……………………………………………………………4p

Subiectul IV(7p)

Se considera sirurile de numere reale nenule $(x_n)_{n\geq 1}$ si $(y_n)_{n\geq 1}$ care verifica relatiile $x_{n+1} = \frac{4+2x_n+x_ny_n}{y_n}$, $y_{n+1}=\frac{4+2y_n+x_ny_n}{x_n}$, $\forall n \in N^*$ si $x_1 = 1, y_1 = 4$.

Sa se calculeze $\lim_{n\to\infty} x_n$ si $\lim_{n\to\infty} y_n$.

Rezolvare: Deoarece $x_1 > 0$ si $y_1 > 0$, prin inductie rezulta $x_n > 0$ *si* $y_n > 0$ pentru orice n≥ 1 ……….……………………………………….. ……………………………………1p

Cele doua siruri sunt strict crescatoare deoarece $x_{n+1} - x_n = \frac{4}{y_n}$+2$\cdot \frac{x_n}{y_n} > 0$ si $y_{n+1} - y_n = \frac{4}{x_n}$+2$\cdot \frac{y_n}{x_n} > 0$,$\forall n \geq 1$ ………………………………………………………………......1p

Avem : $\frac{1}{2+x_{n+1}} - \frac{1}{2+y_{n+1}}$=$\frac{1}{2+x_n} - \frac{1}{2+y_n} = \cdots = \frac{1}{2+x_1} - \frac{1}{2+y_1} = \frac{1}{6}$,$\forall\geq 1$ ……………………...1p

Rezulta ca $\frac{1}{2+x_n} = \frac{1}{6} + \frac{1}{2+y_n} > \frac{1}{6}$,deci $x_n < 4$,$\forall n \geq 1$, de unde $(x_n)_{n\geq 1}$ este convergent....2p

Fie l=$\lim_{n\to\infty} x_n \in (0;4]$ si L=$\lim_{n\to\infty} y_n \in \bar{R}$. Daca L$\in R$, trecand la limita in relatia din enunt obtinem l$\cdot L = 4 + 2l + lL$, deci l=-2 , fals . Ramane L=+∞ si trecand la limita in relatia $\frac{1}{2+x_n} - \frac{1}{2+y_n} = \frac{1}{6}$, obtinem l=4…………………………………………………… 2p

Nota : -problema 4 este din G.M.4/2011 , 26449

-problemele 1 si 2 – prof. Teodor Trisca

- problema 3 , ***