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Olimpiada Națională de Matematică
Etapa locală, 10 februarie 2024
Clasa a IX – a

V1

BAREM ORIENTATIV de CORECTARE și NOTARE:

Problema 1: soluție orientativă

$$M \in (AB) \Rightarrow \overrightarrow{AM} \text{ și } \overrightarrow{AB} \text{ coliniari} \Rightarrow \overrightarrow{AM} = x * \overrightarrow{AB}, x \in \mathbb{R}$$

$$N \in (AC) \Rightarrow \overrightarrow{AN} \text{ și } \overrightarrow{AC} \text{ coliniari} \Rightarrow \overrightarrow{AN} = y * \overrightarrow{AC}, y \in \mathbb{R}$$

1p

$$\overrightarrow{NC} = \overrightarrow{NA} + \overrightarrow{AC} = -y * \overrightarrow{AC} + \overrightarrow{AC} = (1 - y) * \overrightarrow{AC}$$

$$\overrightarrow{MB} = \overrightarrow{MA} + \overrightarrow{AB} = -x * \overrightarrow{AB} + \overrightarrow{AB} = (1 - x) * \overrightarrow{AB}$$

1p

$$\overrightarrow{MD} = \overrightarrow{MB} + \overrightarrow{BD} = (1 - x)\overrightarrow{AB} + \overrightarrow{BD} = (1 - x)\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} = (1 - x)\overrightarrow{AB} + \frac{1}{2}(\overrightarrow{BA} + \overrightarrow{AC}) = (1 - x - \frac{1}{2}) * \overrightarrow{AB} + \frac{1}{2}\overrightarrow{AC} \Rightarrow \overrightarrow{MD} = \frac{1-2x}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AC}$$

1p

$$\overrightarrow{ND} = \overrightarrow{NC} + \overrightarrow{CD} = (1 - y)\overrightarrow{AC} + \frac{1}{2}\overrightarrow{CB} = (1 - y)\overrightarrow{AC} + \frac{1}{2}(\overrightarrow{CA} + \overrightarrow{AB}) = (1 - y - \frac{1}{2}) * \overrightarrow{AC} + \frac{1}{2}\overrightarrow{AB} \Rightarrow$$

$$\overrightarrow{ND} = \frac{1-2y}{2}\overrightarrow{AC} + \frac{1}{2}\overrightarrow{AB}$$

1p

$$\overrightarrow{MB} + \overrightarrow{MD} = (1 - x)\overrightarrow{AB} + \frac{1-2x}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AC} = (\frac{3}{2} - 2x)\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AC}$$

1p

$$\overrightarrow{NC} + \overrightarrow{ND} = (1 - y)\overrightarrow{AC} + \frac{1-2y}{2}\overrightarrow{AC} + \frac{1}{2}\overrightarrow{AB} = (\frac{3}{2} - 2y)\overrightarrow{AC} + \frac{1}{2}\overrightarrow{AB}$$

1p

$$\overrightarrow{MB} + \overrightarrow{MD} \text{ și } \overrightarrow{NC} + \overrightarrow{ND} \text{ sunt coliniari} \Leftrightarrow \frac{\frac{3}{2}-2x}{\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{3}{2}-2y} \Rightarrow (\frac{3}{2} - 2x)(\frac{3}{2} - 2y) = \frac{1}{4}$$

1p

$$\Rightarrow (3 - 4x)(3 - 4y) = 1$$

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Problema 2: soluție orientativă

$$x = \text{card}(A \cap B) \\ \Rightarrow x \leq y < 2^y$$

1p

$$y = \text{card}(A \cup B)$$

Dacă $x = 1 \Rightarrow \frac{1}{y} + \frac{1}{2^y} = \frac{19}{24}$ nu are soluție

Dacă $x = 2 \Rightarrow \frac{1}{2} + \frac{1}{y} + \frac{1}{2^y} = \frac{19}{24} \Rightarrow \frac{1}{y} + \frac{1}{2^y} = \frac{7}{24}$

$y = 2 \Rightarrow \frac{1}{2} + \frac{1}{4} = \frac{7}{24} \Leftrightarrow \frac{3}{4} < \frac{7}{24}$

$y = 3 \Rightarrow \frac{1}{3} + \frac{1}{8} = \frac{11}{24} > \frac{7}{24}$

Pentru $y > 3 \Rightarrow \frac{1}{y} < \frac{1}{3}$

$\Rightarrow \frac{1}{y} + \frac{1}{2^y} < \frac{1}{3} + \frac{1}{8} < \frac{11}{24}$

$2^y > 2^3 \Rightarrow \frac{1}{2^y} < \frac{1}{8}$

Dacă $x = 3 \Rightarrow \frac{1}{3} + \frac{1}{y} + \frac{1}{2^y} = \frac{19}{24} \Rightarrow \frac{1}{y} + \frac{1}{2^y} = \frac{19}{24} - \frac{1}{3} = \frac{11}{24}$

Pentru $y = 3 \Rightarrow \frac{1}{3} + \frac{1}{8} = \frac{11}{24} \Rightarrow y < 3$ convine pentru $x = 3$

Pentru $y > 3 \Rightarrow \frac{1}{y} < \frac{1}{3}$

$\Rightarrow \frac{1}{y} + \frac{1}{2^y} < \frac{1}{3} + \frac{1}{8} = \frac{11}{24}$

$2^y > 2^3 \Rightarrow \frac{1}{2^y} < \frac{1}{8}$

Pentru $y > 3$ ecuația nu are soluții. Analog pt $y < 3$ ecuația nu are soluții.

1p

Dacă $x = 4 \Rightarrow \frac{1}{4} + \frac{1}{y} + \frac{1}{2^y} = \frac{19}{24} \Rightarrow \frac{1}{y} + \frac{1}{2^y} = \frac{19}{24} - \frac{1}{4} = \frac{13}{24}$

$y = 4 \Rightarrow \frac{1}{4} + \frac{1}{16} = \frac{5}{16} < \frac{13}{24}$; Înlocuind $y < 4$ nu se verifică.

$y \geq 5 \Rightarrow \frac{1}{y} \leq \frac{1}{5}$

$\Rightarrow \frac{1}{y} + \frac{1}{2^y} \leq \frac{1}{5} + \frac{1}{32} = \frac{37}{160} < \frac{13}{24}$

$2^y > 2^5 \Rightarrow \frac{1}{2^y} \leq \frac{1}{2^5}$

1p

Dacă $x \geq 4 \Rightarrow \frac{1}{x} \leq \frac{1}{4}$

$4 \leq x \leq y \Rightarrow 4 \leq y \Rightarrow \frac{1}{4} \geq \frac{1}{y} \Rightarrow \frac{1}{y} \leq \frac{1}{4}$
 $2^4 \leq 2^y$

$$\frac{1}{2^4} \geq \frac{1}{2^y} \Leftrightarrow \frac{1}{2^y} \leq \frac{1}{2^4}$$

$$\frac{1}{x} \leq \frac{1}{4}$$

$$\frac{1}{y} \leq \frac{1}{4} \Rightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{2^y} \leq \frac{1}{4} + \frac{1}{4} + \frac{1}{2^4} = \frac{9}{16} < \frac{19}{24}$$
$$\frac{1}{2^y} \leq \frac{1}{2^4}$$

1p

Soluția convenabila $x = 3$ și $y = 3 \Rightarrow \text{card}(A \cup B) = \text{card}(A \cap B) \Rightarrow \text{card}(A) = \text{card}(B)$.

1p

Problema 3: soluție orientativă

$$\left\{ \frac{4x+3}{3x-2} \right\} = \frac{4x+3}{3x-2} - \left[\frac{4x+3}{3x-2} \right] = \frac{2x-2}{3x-2}$$

1p

$$\left[\frac{4x+3}{3x-2} \right] = \frac{4x+3}{3x-2} - \frac{2x-2}{3x-2} = \frac{2x+5}{3x-2}$$

1p

$$\frac{2x+5}{3x-2} = k, k \in \mathbb{Z};$$

$$2x+5 = k \cdot (3x-2) \Rightarrow 2x+5-3kx+2k=0 \Rightarrow x(2-3k)+5+2k=0 \Rightarrow$$

$$\Rightarrow x = \frac{2k+5}{3k-2}$$

1p

$$\left[\frac{4 \frac{2k+5}{3k-2} + 3}{3 \frac{2k+5}{3k-2} - 2} \right] = k \Rightarrow \left[\frac{\frac{8k+20+9k-6}{3k-2}}{\frac{6k+15-6k+4}{3k-2}} \right] = k \Rightarrow \left[\frac{\frac{17k+14}{3k-2}}{\frac{19}{3k-2}} \right] = k \Rightarrow \left[\frac{17k+14}{19} \right] = k \Rightarrow$$

1p

$$\Rightarrow k \leq \frac{17k+14}{19} < k+1$$

1p

$$19k \leq 17k+14 \Rightarrow 2k \leq 14 \Rightarrow k \leq 7$$

$$\Rightarrow k \in \left(-\frac{5}{2}, 7 \right]$$

$$17k+14 < 19k+19 \Rightarrow -2k+5 \Rightarrow k > \frac{-5}{2}$$

1p

$$k \in \mathbb{Z} \Rightarrow k \in \{-2, -1, 0, 1, 2, 3, 4, 5, 6, 7\}$$

$$x \in \left\{ -\frac{1}{8}, -\frac{3}{5}, -\frac{5}{2}, 7, \frac{9}{4}, \frac{11}{7}, \frac{13}{10}, \frac{15}{13}, \frac{17}{16}, 1 \right\}$$

1p

Problema 4: soluție orientativă

$$\frac{bc}{b+c} = \frac{2abc}{2a(b+c)}, \frac{ab}{a+b} = \frac{2abc}{2c(a+b)}, \frac{ac}{a+c} = \frac{2abc}{2b(a+c)}$$

1p

Aplicăm inegalitatea mediilor pentru $2a$ și $b+c$ și obținem:

$$\frac{bc}{b+c} = \frac{2abc}{2a(b+c)} \geq \frac{2abc}{\left(\frac{2a+b+c}{2}\right)^2} = \frac{8abc}{(2a+b+c)^2}$$

1p

Scriem omoloagele:

$$\frac{ab}{a+b} = \frac{2abc}{2c(a+b)} \geq \frac{2abc}{\left(\frac{2c+b+a}{2}\right)^2} = \frac{8abc}{(a+b+2c)^2}$$

$$\frac{ac}{a+c} = \frac{2abc}{2b(a+c)} \geq \frac{2abc}{\left(\frac{2b+a+c}{2}\right)^2} = \frac{8abc}{(a+2b+c)^2}$$

Insumăm inegalitățile și obținem:

$$\frac{bc}{b+c} + \frac{ab}{a+b} + \frac{ac}{a+c} \geq 8abc \left(\frac{1}{(2a+b+c)^2} + \frac{1}{(a+2b+c)^2} + \frac{1}{(a+b+2c)^2} \right)$$

1p

Aplicăm inegalitatea:

$$3(x^2 + y^2 + z^2) \geq (x + y + z)^2$$

$$\text{pentru } x = \frac{1}{2a+b+c}, y = \frac{1}{a+b+2c}, z = \frac{1}{a+2b+c}$$

1p

$$\frac{bc}{b+c} + \frac{ac}{a+c} + \frac{ab}{a+b} \geq 8abc \left(\frac{1}{(2a+b+c)^2} + \frac{1}{(a+2b+c)^2} + \frac{1}{(a+b+2c)^2} \right)$$

$$\geq 8abc \frac{1}{3} \left(\frac{1}{2a+b+c} + \frac{1}{a+2b+c} + \frac{1}{a+b+2c} \right)^2$$

1p

Aplicăm inegalitatea *Titu Andreescu* pentru:

$$a_1 = 1, a_2 = 1, a_3 = 1$$

$$x_1 = 2a + b + c$$

$$x_2 = a + 2b + c$$

$$x_3 = a + b + 2c$$

1p

Și obținem:

$$8abc \frac{1}{3} \left(\frac{1^2}{2a+b+c} + \frac{1^2}{a+2b+c} + \frac{1^2}{a+b+2c} \right)^2 \geq 8abc \frac{1}{3} \left[\frac{(1+1+1)^2}{4(a+b+c)} \right]^2 =$$

$$= 8abc \frac{1}{3} \left(\frac{3^2}{4(a+b+c)} \right)^2 = 8abc \cdot \frac{1}{3} \cdot \frac{81}{16} = \frac{27abc}{2}$$

1p

Notă: Orice altă soluție corectă se punctează corespunzător