Soluţii clasa a IX-a:

1.a) Deoarece $x \in (0,1) \Rightarrow x < \sqrt{x} < 1$(*). Considerăm $x = 0,\underbrace{99\ldots9}_{4032\text{ ori}} = 1 - \frac{1}{10^{4032}}$. Folosind relația (*), deducem că primele 2016 zecimale ale lui $\sqrt{x}$ vor fi egale cu 9. Cum $d_{4032} = \frac{x}{9}, \sqrt{d_{4032}} = \frac{\sqrt{x}}{3} = 0,\underbrace{33\ldots3}_{2016\text{ ori}}\ldots$. În concluzie primele 2016 zecimale ale numărului $\sqrt{d_{4032}}$ sunt egale cu 3.

b) $\sum_{k=1}^{n} 10^{k} \cdot d_k = 1 + 11 + 111 + \cdots + \underbrace{111\ldots1}_{n\text{ ori}} =$

$= \frac{1}{9}\cdot(10-1) + \frac{1}{9}\cdot(10^2-1) + \ldots + \frac{1}{9}\cdot(10^n-1) = \frac{1}{9}\cdot\left(\frac{10\cdot(10^n-1)}{10-1} - n\right) =$

$= \frac{10}{81}\cdot(10^n-1) - \frac{n}{9}$.

2. Din $a+b=2 \Rightarrow 1 = \frac{a+b}{2} \geq \sqrt{ab}$, de unde rezultă $\sqrt{ab} \leq 1$.

Prin ridicare la pătrat a inegalității date obținem:

$$x + \frac{1}{a^2} + x + \frac{1}{b^2} + 2\sqrt{\left(x+\frac{1}{a^2}\right)\left(x+\frac{1}{b^2}\right)} \geq 4x + 4.$$

Dar $\frac{1}{a^2} + \frac{1}{b^2} \geq 2\sqrt{\frac{1}{a^2}\cdot\frac{1}{b^2}} = \frac{2}{ab} \geq 2$ (1) și $\sqrt{\left(x+\frac{1}{a^2}\right)\left(x+\frac{1}{b^2}\right)} = \sqrt{x^2 + x\left(\frac{1}{a^2}+\frac{1}{b^2}\right) + \frac{1}{a^2b^2}} \Rightarrow$

$\Rightarrow \sqrt{x^2 + x\left(\frac{1}{a^2}+\frac{1}{b^2}\right) + \frac{1}{a^2b^2}} \geq \sqrt{x^2+2x+1} \Rightarrow \sqrt{\left(x+\frac{1}{a^2}\right)\left(x+\frac{1}{b^2}\right)} \geq x+1$ (2).

Din relațiile (1) și (2) rezultă inegalitatea cerută.

3. $ABCD$ este paralelogram $\Leftrightarrow \overrightarrow{CD} = \overrightarrow{BA}$. Fie $\overrightarrow{AB} = \vec{u}, \overrightarrow{BC} = \vec{v}, \overrightarrow{CD} = a\vec{u} + b\vec{v}$, cu a şi b numere reale. Rezultă:

$\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = (a+1)\vec{u} + (b+1)\vec{v},$

$\overrightarrow{AN} = \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} = \vec{u} + \frac{1}{2}\vec{v}, \overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD} = a\vec{u} + (b+1)\vec{v}, \overrightarrow{BE} = \frac{1}{3}\overrightarrow{BD} = \frac{a}{3}\vec{u} + \frac{b+1}{3}\vec{v},$

$\overrightarrow{AE} = \overrightarrow{AB} + \overrightarrow{BE} = \frac{a+3}{3}\vec{u} + \frac{b+1}{3}\vec{v};$

$\overrightarrow{DF} = \overrightarrow{DA} + \overrightarrow{AF} = -(a+1)\vec{u} - (b+1)\vec{v} + \frac{1}{3}(\vec{u}+\vec{v}) = -(a+\frac{2}{3})\vec{u} - (b+\frac{2}{3})\vec{v},$

$\overrightarrow{DM} = \overrightarrow{DA} + \overrightarrow{AM} = -(a+1)\vec{u} - (b+1)\vec{v} + \frac{1}{2}\vec{u} = -(a+\frac{1}{2})\vec{u} - (b+1)\vec{v}$.

Din $\overrightarrow{AE}, \overrightarrow{AN}$ coliniari deducem: $a - 2b + 1 = 0$. Analog din $\overrightarrow{DF}, \overrightarrow{DM}$ coliniari urmează că $2a + b + 2 = 0$. Rezultă $a = -1$; $b = 0$. În concluzie, $\overrightarrow{CD} = \overrightarrow{BA}$.

4. Pentru orice punct X din plan notăm cu $\overrightarrow{r_X}$ vectorul de poziție asociat, iar cu $\vec{r}=\overrightarrow{r_A}+\overrightarrow{r_B}+\overrightarrow{r_C}+\overrightarrow{r_D}+\overrightarrow{r_E}+\overrightarrow{r_F}$. Avem:

$$\overrightarrow{r_{G_A}}=\frac{\vec{r}-\overrightarrow{r_A}}{5},\overrightarrow{r_{G_B}}=\frac{\vec{r}-\overrightarrow{r_B}}{5},\overrightarrow{r_{G_C}}=\frac{\vec{r}-\overrightarrow{r_C}}{5},\overrightarrow{r_{G_D}}=\frac{\vec{r}-\overrightarrow{r_D}}{5},\overrightarrow{r_{G_E}}=\frac{\vec{r}-\overrightarrow{r_E}}{5},\overrightarrow{r_{G_F}}=\frac{\vec{r}-\overrightarrow{r_F}}{5}\ .$$

a)Cu vectorii $\overrightarrow{AG_A},\overrightarrow{BG_B},\overrightarrow{CG_C},\overrightarrow{DG_D},\overrightarrow{EG_E},\overrightarrow{FG_F}$ se poate construi un hexagon dacă și numai dacă $\overrightarrow{AG_A}+\overrightarrow{BG_B}+\overrightarrow{CG_C}+\overrightarrow{DG_D}+\overrightarrow{EG_E}+\overrightarrow{FG_F}=\vec{0}$.

$$\overrightarrow{AG_A}+\overrightarrow{BG_B}+\overrightarrow{CG_C}+\overrightarrow{DG_D}+\overrightarrow{EG_E}+\overrightarrow{FG_F}=\vec{0}$$

$$=\left(\overrightarrow{r_{G_A}}-\overrightarrow{r_A}\right)+\left(\overrightarrow{r_{G_B}}-\overrightarrow{r_B}\right)+\left(\overrightarrow{r_{G_C}}-\overrightarrow{r_C}\right)+\left(\overrightarrow{r_{G_D}}-\overrightarrow{r_D}\right)+\left(\overrightarrow{r_{G_E}}-\overrightarrow{r_E}\right)+\left(\overrightarrow{r_{G_F}}-\overrightarrow{r_F}\right)=$$

$$=\left(\frac{\vec{r}-\overrightarrow{r_A}}{5}-\overrightarrow{r_A}\right)+\left(\frac{\vec{r}-\overrightarrow{r_B}}{5}-\overrightarrow{r_B}\right)+\left(\frac{\vec{r}-\overrightarrow{r_C}}{5}-\overrightarrow{r_C}\right)+\left(\frac{\vec{r}-\overrightarrow{r_D}}{5}-\overrightarrow{r_D}\right)+\left(\frac{\vec{r}-\overrightarrow{r_E}}{5}-\overrightarrow{r_E}\right)+\left(\frac{\vec{r}-\overrightarrow{r_F}}{5}-\overrightarrow{r_F}\right)=$$

$$=\frac{6}{5}\left(\vec{r}-\left(\overrightarrow{r_A}+\overrightarrow{r_B}+\overrightarrow{r_C}+\overrightarrow{r_D}+\overrightarrow{r_E}+\overrightarrow{r_F}\right)\right)=\frac{6}{5}\left(\vec{r}-\vec{r}\right)=\vec{0}.$$

b) $\overrightarrow{G_AG_B}=\overrightarrow{r_{G_B}}-\overrightarrow{r_{G_A}}=\frac{\vec{r}-\overrightarrow{r_B}}{5}-\frac{\vec{r}-\overrightarrow{r_A}}{5}=\frac{\overrightarrow{r_A}-\overrightarrow{r_B}}{5}=\frac{1}{5}\overrightarrow{BA}$. Analog:

$$\overrightarrow{G_BG_C}=\frac{1}{5}\overrightarrow{CB},\overrightarrow{G_CG_D}=\frac{1}{5}\overrightarrow{DC},\overrightarrow{G_DG_E}=\frac{1}{5}\overrightarrow{ED},\overrightarrow{G_EG_F}=\frac{1}{5}\overrightarrow{FE},\overrightarrow{G_FG_A}=\frac{1}{5}\overrightarrow{AF}\ .$$

Hexagoanele $G_AG_BG_CG_DG_EG_F$ și $ABCDEF$ au laturile paralele, deci sunt asemenea, raportul de asemănare fiind $\frac{1}{5}\Rightarrow\frac{A_{[G_AG_BG_CG_DG_EG]}}{A_{[ABCDEF]}}=\left(\frac{1}{5}\right)^2=\frac{1}{25}$.