OLIMPIADA NAŢIONALĂ DE MATEMATICĂ

Faza locală-14.02.2014

Clasa a IX-a

Soluţii

1) Presupunem că există cel puțin doi termeni $a_m, a_n \in \mathbb{Q}$.Rezultă că $a_m - a_n = (m-n)\cdot r \in \mathbb{Q}$ Cum $m \neq n \Rightarrow r \in \mathbb{Q}$. Prin urmare $a_m = a_1 + (m-1)r \in \mathbb{R} - \mathbb{Q}$. Contradicție!

2) a) $P(n): \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + ... + \frac{1}{(2n-1)(2n+1)} = \frac{n-1}{3(2n+1)}, n \geq 2$. $P(2): \frac{1}{15} = \frac{1}{15}$ adevărată.

$P(k) \to P(k+1)$

$$\frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + ... + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2k+1)(2k+3)} = \frac{k-1}{3(2k+1)} + \frac{1}{(2k+1)(2k+3)} =$$

$= \frac{2k^2 + 3k - 2k - 3 + 3}{3(2k+1)(2k+3)} = \frac{2k^2 + k}{3(2k+1)(2k+3)} = \frac{k}{3(2k+3)}$. Deci P(n) este adevărată $\forall n \geq 2$.

b) $[S_n] = \left[\frac{n-1}{3(2n+1)}\right] = 0$, deci $\{S_n\} = S_n \Rightarrow \left\{\frac{n-1}{3(2n+1)}\right\} = \frac{n-1}{3(2n+1)}$.Deci $\frac{n-1}{3(2n+1)} \in \left[\frac{1}{15}, \frac{1}{7}\right]$

$\Leftrightarrow \frac{1}{15} \leq \frac{n-1}{3(2n+1)} \leq \frac{1}{7} \Leftrightarrow 7n - 7 \leq 6n + 3 \leq 15n - 15 \Leftrightarrow 2 \leq n \leq 10$ rezultă $n \in \{2;3;...;10\}$.

3) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DA} = \vec{0} \Rightarrow \overrightarrow{AB} + \overrightarrow{CD} = \overrightarrow{AD} + \overrightarrow{CB}$

$$2\overrightarrow{EF} = \overrightarrow{EB} + \overrightarrow{ED} = -\frac{1}{2}\left(\overrightarrow{BC} + \overrightarrow{BA}\right) - \frac{1}{2}\left(\overrightarrow{DC} + \overrightarrow{DA}\right) = \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{CD} + \overrightarrow{AD} + \overrightarrow{CB}\right) = \overrightarrow{AB} + \overrightarrow{CD}$$

4) a) $\overrightarrow{AM} = k\overrightarrow{MD}, \overrightarrow{BN} = k\overrightarrow{NC}$ și $k > 0 \Rightarrow M \in [AD], N \in [BC]$

$\overrightarrow{PM} = \overrightarrow{PA} + \overrightarrow{AM} = -\frac{1}{2}\overrightarrow{AB} + \frac{k}{k+1}\overrightarrow{AD}$, $\overrightarrow{PN} = \overrightarrow{PB} + \overrightarrow{BN} = \frac{1}{2}\overrightarrow{AB} + \frac{k}{k+1}\overrightarrow{BC}$, deci

$\overrightarrow{PR} = \frac{1}{2}(\overrightarrow{PM} + \overrightarrow{PN}) = \frac{k}{2(k+1)}(\overrightarrow{AD} + \overrightarrow{BC})$. Dar

$\overrightarrow{PQ} = \frac{1}{2}(\overrightarrow{PC} + \overrightarrow{PD}) = \frac{1}{2}(\overrightarrow{PB} + \overrightarrow{BC} + + \overrightarrow{PA} + \overrightarrow{AD}) = \frac{1}{2}(\overrightarrow{AD} + \overrightarrow{BC})$,deci $\overrightarrow{PR} = \frac{k}{k+1}\overrightarrow{PQ}$,de unde rezultă coliniaritatea punctelor *P*,*Q*,*R*.

c) Patrulaterul *MPNQ* paralelogram $\Leftrightarrow \overrightarrow{PR} = \frac{1}{2}\overrightarrow{PQ} \Leftrightarrow \frac{k}{k+1} = \frac{1}{2} \Leftrightarrow k = 1$.

OLIMPIADA NAŢIONALĂ DE MATEMATICĂ

Faza locală-15.02.2014

Clasa a IX-a

Bareme

1) Presupunem că există cel puțin doi termeni $a_m, a_n \in \mathbb{Q}$ ……………………………...….2p

Rezultă că $a_m - a_n = (m-n)\cdot r \in \mathbb{Q}$ ………………………………………………...….2p

Cum $m \neq n \Rightarrow r \in \mathbb{Q}$ ………………………………………………………………….….1p

Prin urmare $a_m = a_1 + (m-1)r \in \mathbb{R} - \mathbb{Q}$. Contradicție!..2p

2) a) $P(n): \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + ... + \frac{1}{(2n-1)(2n+1)} = \frac{n-1}{3(2n+1)}, n \geq 2$.verificare P(2)…………….....1p

Demonstrarea implicației $P(k) \to P(k+1)$…Deci P(n) este adevărată $\forall n \geq 2$ ………….2p

b) $[S_n] = \left[\frac{n-1}{3(2n+1)}\right] = 0$, deci $\{S_n\} = S_n \Rightarrow \left\{\frac{n-1}{3(2n+1)}\right\} = \frac{n-1}{3(2n+1)}$ ……………………..2p

$\frac{n-1}{3(2n+1)} \in \left[\frac{1}{15}, \frac{1}{7}\right] \Leftrightarrow \frac{1}{15} \leq \frac{n-1}{3(2n+1)} \leq \frac{1}{7} \Leftrightarrow 2 \leq n \leq 10$,deci $n \in \{2;3;...;10\}$ …………………….2p

3) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DA} = \vec{0}$ ………………………………………………………………1p

$\Rightarrow \overrightarrow{AB} + \overrightarrow{CD} = \overrightarrow{AD} + \overrightarrow{CB}$ ……………………………………………………………...….2p

$2\overrightarrow{EF} = \overrightarrow{EB} + \overrightarrow{ED} = -\frac{1}{2}\left(\overrightarrow{BC} + \overrightarrow{BA}\right) - \frac{1}{2}\left(\overrightarrow{DC} + \overrightarrow{DA}\right) = \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{CD} + \overrightarrow{AD} + \overrightarrow{CB}\right) = \overrightarrow{AB} + \overrightarrow{CD}$ ……4p

4) a) $\overrightarrow{AM} = k\overrightarrow{MD}, \overrightarrow{BN} = k\overrightarrow{NC}$ și $k > 0 \Rightarrow M \in [AD], N \in [BC]$ …...…………………………1p

$\overrightarrow{PM} = \overrightarrow{PA} + \overrightarrow{AM} = -\frac{1}{2}\overrightarrow{AB} + \frac{k}{k+1}\overrightarrow{AD}$,…....……………………………………………………1p

$\overrightarrow{PN} = \overrightarrow{PB} + \overrightarrow{BN} = \frac{1}{2}\overrightarrow{AB} + \frac{k}{k+1}\overrightarrow{BC}$ ……………………………………………………………1p

$\overrightarrow{PR} = \frac{1}{2}(\overrightarrow{PM} + \overrightarrow{PN}) = \frac{k}{2(k+1)}(\overrightarrow{AD} + \overrightarrow{BC})$ ……………………………………………………..1p.

$\overrightarrow{PQ} = \frac{1}{2}(\overrightarrow{PC} + \overrightarrow{PD}) = \frac{1}{2}(\overrightarrow{PB} + \overrightarrow{BC} + +\overrightarrow{PA} + \overrightarrow{AD}) = \frac{1}{2}(\overrightarrow{AD} + \overrightarrow{BC})$ ………………………………….1p

deci $\overrightarrow{PR} = \frac{k}{k+1}\overrightarrow{PQ}$,de unde rezultă coliniaritatea punctelor P,Q,R……………..………………1p

b) Patrulaterul $MPNQ$ paralelogram $\Leftrightarrow \overrightarrow{PR} = \frac{1}{2}\overrightarrow{PQ} \Leftrightarrow \frac{k}{k+1} = \frac{1}{2} \Leftrightarrow k = 1$ ……………………1p